CHAPTER 5

The Differential Forms of the Fundamental Laws

5.1 $0 = \int_{c.v.} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{c.s.} \rho \vec{V} \cdot \hat{n} dA.$ Using Gauss' theorem:

$$0 = \int_{c.v.} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{c.v.} \vec{\nabla} \cdot (\rho \vec{V}) d\mathcal{V} = \int_{c.v.} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) \right] d\mathcal{V}.$$

Since this is true for all arbitrary control volumes (i.e., for all limits of integration), the integrand must be zero:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0.$$

This can be written in rectangular coordinates as

$$-\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w).$$

This is Eq. 5.2.2. The other forms of the continuity equation follow.

5.2 $\dot{m}_{in} - \dot{m}_{out} = \frac{\partial m_{element}}{\partial t}.$

$$\rho v_r (r d\theta dz) - \left[\rho v_r + \frac{\partial}{\partial r}(\rho v_r) dr \right] (r + dr) d\theta dz$$

$$+ \rho v_\theta dr dz - \left[\rho v_\theta + \frac{\partial}{\partial \theta}(\rho v_\theta) d\theta \right] dr dz$$

$$+ \rho v_z \left(r + \frac{dr}{2} \right) d\theta dr - \left[\rho v_z + \frac{\partial}{\partial z}(\rho v_z) dz \right] \left(r + \frac{dr}{2} \right) d\theta dr = \frac{\partial}{\partial t} \left[\rho \left(r + \frac{dr}{2} \right) d\theta dr dz \right].$$

Subtract terms and divide by $rdqdrdz$:

$$-\frac{\rho v_r}{r} - \frac{\partial}{\partial r}(\rho v_r) \frac{r + dr}{r} - \frac{\partial}{\partial \theta}(\rho v_\theta) \frac{1}{r} - \frac{\partial}{\partial z}(\rho v_z) \frac{r + dr\ /2}{r} = \frac{\partial}{\partial t} \rho \frac{r + dr\ /2}{r}.$$

Since dr is an infinitesimal, $(r + dr)/\ r = 1$ and $(r + dr/\ 2)/\ r = 1$. Hence,

$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) + \frac{1}{r}\rho v_r = 0.$ This can be put in various forms.

5.3 $\dot{m}_{in} - \dot{m}_{out} = \dfrac{\partial m_{element}}{\partial t}.$

$$\rho v_r (rd\theta) r \sin\theta d\phi - \left[\rho v_r + \frac{\partial}{\partial r}(\rho v_r) dr\right](r+dr)d\theta(r+dr)\sin\theta d\phi$$

$$+\rho v_\theta dr\left(r + \frac{dr}{2}\right)\sin\theta d\phi - \left[\rho v_\theta + \frac{\partial}{\partial\theta}(\rho v_\theta)d\theta\right]dr\left(r+\frac{dr}{2}\right)\sin\theta d\phi$$

$$+\rho v_\phi dr\left(r+\frac{dr}{2}\right)d\theta - \left[\rho v_\phi + \frac{\partial}{\partial\phi}(\rho v_\phi)d\phi\right]dr\left(r+\frac{dr}{2}\right)d\theta$$

$$= \frac{\partial}{\partial t}\left[\rho\left(r+\frac{dr}{2}\right)^2 drd\theta \sin\theta d\phi\right]$$

Because some areas are not rectangular, we used an average length $(r + dr/2)$. Now, subtract some terms and divide by $rdqdfdr$:

$$-\rho v_r \sin\theta - \rho v_r \sin\theta - \frac{\partial}{\partial r}(\rho v_r)\sin\theta \frac{(r+dr)^2}{r} - \frac{\partial}{\partial\theta}(\rho v_\theta)\frac{r+\frac{dr}{2}}{r}\sin\theta$$

$$-\frac{\partial}{\partial\phi}(\rho v_\phi)\frac{r+\frac{dr}{2}}{r} = \frac{\partial\rho}{\partial t}\frac{\left(r+\frac{dr}{2}\right)^2}{r}\sin\theta$$

Since dr is infinitesimal $(r+dr)^2/r = r$ and $(r+dr/2)/r = 1.$ Divide by $r\sin q$ and there results

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho v_\phi) + \frac{2}{r}\rho v_r = 0$$

5.4 For a steady flow $\dfrac{\partial\rho}{\partial t} = 0.$ Then, with $v = w = 0$ Eq. 5.2.2 yields

$$\frac{\partial}{\partial x}(\rho u) = 0 \qquad \text{or} \qquad \underline{\rho\frac{du}{dx} + u\frac{d\rho}{dx} = 0.}$$

Partial derivatives are not used since there is only one independent variable.

5.5 Since the flow is incompressible $\dfrac{Dr}{Dt} = 0.$ This gives

$$\therefore \nabla p = \frac{\P p}{\P r}\hat{i}_r + \frac{1}{r}\frac{\P p}{\P q}\hat{i}_q = \underline{\frac{200r}{r^3}\left[\frac{1}{r^2} - \cos 2q\right]\hat{i}_r - \frac{200r}{r^3}\sin 2q\,\hat{i}_q} \qquad \text{or}$$

$$\underline{u\frac{\partial\rho}{\partial x} + w\frac{\partial\rho}{\partial z} = 0.}$$

Also, $\vec{\nabla}\cdot\vec{V} = 0,$ or $\underline{\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0.}$

5.6 Given: $\frac{\partial}{\partial t} = 0,\ \frac{\partial \rho}{\partial z} \neq 0$. Since water can be considered to be incompressible, we demand that $\frac{Dr}{Dt} = 0$. $\therefore\ \underline{u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z}} = 0,$ assuming the x-direction to be in the direction of flow. Also, we demand that $\vec{\nabla} \cdot \vec{V} = 0,$ or $\underline{\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.}$

5.7 We can use the ideal gas law, $r = \frac{p}{RT}$. Then, the continuity equation $\frac{Dr}{Dt} = -r\vec{\nabla} \cdot \vec{V}$ becomes, assuming RT to be constant, $\frac{1}{RT}\frac{Dp}{Dt} = -\frac{p}{RT}\vec{\nabla} \cdot \vec{V}$ or

$$\frac{1}{p}\frac{Dp}{Dt} = -\vec{\nabla} \cdot \vec{V}.$$

5.8 a) Use cylindrical coordinates with $v_q = v_z = 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0$$

Integrate:

$$rv_r = C. \qquad \therefore\ \underline{v_r = \frac{C}{r}}.$$

b) Use spherical coordinates with $v_q = v_f = 0$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = 0$$

Integrate:

$$r^2 v_r = C. \qquad \therefore\ \underline{v_r = \frac{C}{r^2}}.$$

5.9 $$\frac{D\rho}{Dt} = -\rho\vec{\nabla} \cdot \vec{V} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -2.3(200 \times 1 + 400 \times 1) = \underline{-1380\ \frac{\text{kg}}{\text{m}^3 \cdot \text{s}}}.$$

5.10 In a plane flow, $u = u(x, y)$ and $v = v(x, y)$. Continuity demands that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. If $u = \text{const}$, then $\frac{\partial u}{\partial x} = 0$ and hence $\frac{\partial v}{\partial y} = 0$. Thus, $\underline{v = \text{const}}$ also.

5.11 If $u = C_1$ and $v = C_2$, the continuity equation provides, for an incompressible flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad \therefore \frac{\partial w}{\partial z} = 0 \text{ and } w = C_3.$$

The z-component of velocity $\underline{w \text{ is also constant}}$.

We also have

$$\frac{D\rho}{Dt} = 0 = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

The density may vary with x, y, z and t. It is not, necessarily, constant.

5.12 $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad \therefore\ A + \frac{\partial v}{\partial y} = 0. \qquad \therefore\ v(x,y) = -Ay + f(x).$

But, $v(x,o) = 0 = f(x). \qquad \therefore\ \underline{v = -Ay}.$

5.13 $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad \therefore \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{(x^2+y^2)5 - 5x(2x)}{(x^2+y^2)^2} = -\frac{5x^2 - 5y^2}{(x^2+y^2)^2}$

$$\therefore v(x,y) = \int \frac{5y^2 - 5x^2}{(x^2+y^2)^2}dy + f(x) = \frac{5y}{x^2+y^2} + f(x). \quad f(x) = 0. \quad \therefore v = \underline{\frac{5y}{x^2+y^2}}.$$

5.14 From Table 5.1: $\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = -\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = -\frac{1}{r}\left(10 + \frac{.4}{r^2}\right)\sin\theta.$

$$\therefore rv_r = \int\left(10 + \frac{.4}{r^2}\right)\sin q\, dr + f(q) = \left(10r - \frac{.4}{r}\right)\sin q + f(q).$$

$$.2v_r(.2,q) = \left(10\times .2 - \frac{.4}{.2}\right)\sin q + f(q) = 0. \qquad \therefore f(q) = 0.$$

$$\therefore v_r = \underline{\left(10 - \frac{0.4}{r^2}\right)\sin q}.$$

5.15 From Table 5.1: $\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = -\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = \frac{-20}{r}\left(1 + \frac{1}{r^2}\right)\cos\theta.$

$$\therefore rv_r = \int -20\left(1 + \frac{1}{r^2}\right)\cos q\, dr + f(q) = -20\left(r - \frac{1}{r}\right)\cos q + f(q).$$

$$v_r(1,q) = -20(1-1)\cos q + f(q) = 0. \qquad \therefore f(q) = 0.$$

$$\therefore v_r = \underline{-20\left(1 - \frac{1}{r^2}\right)\cos q}.$$

5.16 From Table 5.1, spherical coordinates: $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta)$.

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r\sin\theta}\left(10+\frac{40}{r^3}\right)2\sin\theta\cos\theta.$$

$$\therefore r^2 v_r = \int r\left(10+\frac{40}{r^3}\right)2\cos q\,dr + f(q) = \left(10r^2 - \frac{80}{r}\right)\cos q + f(q)$$

$$4v_r(2,q) = \left(10\times 2^2 - \frac{80}{2}\right)\cos q + f(q) = 0. \qquad \therefore f(q) = 0.$$

$$\therefore v_r = \underline{\left(10-\frac{80}{r^3}\right)\cos q}.$$

5.17 Continuity: $\frac{\partial}{\partial x}(\rho u) = 0. \qquad \therefore \rho\frac{du}{dx} + u\frac{d\rho}{dx} = 0.$

$$r = \frac{p}{RT} = \frac{18\times 144}{1716\times 500} = 0.00302\ \frac{\text{slug}}{\text{ft}^3}. \quad \frac{du}{dx} = \frac{526-453}{2\times 2/\ 12} = 219\text{ fps}/\ \text{ft}.$$

$$\therefore \frac{dr}{dx} = -\frac{r}{u}\frac{du}{dx} = -\frac{.00302}{486}\times 219 = \underline{-0.00136\text{ slug}/\ \text{ft}^4}.$$

5.18 $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad \frac{\partial}{\partial x}\left[-20(1-e^{-x})\right] = -20e^{-x}$

Hence, in the vicinity of the x-axis:

$$\frac{\partial v}{\partial y} = 20e^{-x} \text{ and } v = 20ye^{-x} + C.$$

But $v = 0$ if $y = 0$. $\quad \therefore C = 0.$

$$v = 20ye^{-x} = 20(0.2)e^{-2} = \underline{0.541\text{ m}/\ \text{s}}$$

5.19 $\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0. \qquad \frac{\partial}{\partial z}\left[-20(1-e^{-z})\right] = -20e^{-z}$

Hence, in the vicinity of the z-axis:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 20e^{-z} \text{ and } rv_r = \frac{r^2}{2}20e^{-z} + C.$$

But $v_r = 0$ if $r = 0$. $\quad \therefore C = 0.$

$$v_r = 10re^{-z} = 10(0.2)e^{-2} = \underline{0.271\text{ m}/\ \text{s}}$$

5.20 The velocity is zero at the stagnation point. Hence,

$$0 = 10 - \frac{40}{R^2}. \qquad \therefore \underline{R = 2\text{ m}}$$

The continuity equation for this plane flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. Using $\frac{\partial u}{\partial x} = 80x^{-3}$, we see that $\frac{\partial v}{\partial y} = -80x^{-3}$ near the x-axis. Consequently, for small Δy,

$$\Delta v = -80x^{-3}\Delta y \quad \text{so that} \quad v = -80(-3)^{-3}(0.1) = \underline{0.296 \text{ m/s}}.$$

5.21 The velocity is zero at the stagnation point. Hence

$$0 = \frac{40}{R^2} - 10. \qquad \therefore \underline{R = 2 \text{ m}}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(40 - 10r^2) = -\frac{20}{r}.$$

Near the negative x-axis continuity provides us with

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta \sin\theta\right) = \frac{20}{r}.$$

Integrate, letting $q = 0$ from the y-axis:

$$v_q \sin q = -20\cos q + C$$

Since $v_q = 0$ when $q = 90^\circ$, $C = 0$. Then, with $a = \tan^{-1}\frac{0.1}{3} = 1.909^\circ$,

$$v_q = -20\frac{\cos q}{\sin q} = -20\frac{\cos 88.091}{\sin 88.091} = -20\frac{0.0333}{0.999} = \underline{0.667 \text{ m/s}}$$

5.22 Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad \therefore \frac{\Delta v}{\Delta y} = -\frac{\Delta u}{\Delta x} = -\frac{13.5 - 11.3}{2\times.005} = -220 \ \frac{\text{m/s}}{\text{m}}.$

$\therefore \Delta v = v - 0 = -220\Delta y. \qquad \therefore v = -220\times.004 = \underline{-0.88 \text{ m/s}}.$

b) $a_x = u\frac{\partial u}{\partial x} = 12.6\times(+220) = \underline{2772 \text{ m/s}^2}.$

5.23 $\Sigma F_y = ma_y$. For the fluid particle occupying the volume of Fig. 5.3:

$$\left(\tau_{yy} + \frac{\partial\tau_{yy}}{\partial y}\frac{dy}{2}\right)dxdz + \left(\tau_{zy} + \frac{\partial\tau_{zy}}{\partial z}\frac{dz}{2}\right)dxdy + \left(\tau_{xy} + \frac{\partial\tau_{xy}}{\partial x}\frac{dx}{2}\right)dydz$$

$$-\left(\tau_{yy} - \frac{\partial\tau_{yy}}{\partial y}\frac{dy}{2}\right)dxdz - \left(\tau_{zy} - \frac{\partial\tau_{zy}}{\partial z}\frac{dz}{2}\right)dxdy - \left(\tau_{xy} - \frac{\partial\tau_{xy}}{\partial x}\frac{dx}{2}\right)dydz$$

$$+rg_y\,dx\,dy\,dz = r dx\,dy\,dz\frac{Dv}{Dt}$$

Dividing by $dx\,dy\,dz$, and adding and subtracting terms:

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + \rho g_y = \rho\frac{Dv}{Dt}.$$

5.24 Check continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{(x^2 + y^2)10 - 10x(2x)}{(x^2 + y^2)^2} + \frac{(x^2 + y^2)10 - 10y(2y)}{(x^2 + y^2)^2} = 0.$$

Thus, it is a possible flow. For a frictionless flow, Euler's Eqs. 5.3.7 give, with $g_x = g_y = 0$:

$$r u \frac{\P u}{\P x} + r v \frac{\P u}{\P y} = -\frac{\P p}{\P x}.$$

$$\therefore \frac{\P p}{\P x} = -r \frac{10x}{x^2+y^2} \frac{10y^2 - 10x^2}{(x^2+y^2)^2} - r \frac{10y}{x^2+y^2} \frac{-20xy}{(x^2+y^2)^2} = r \frac{100(x^2+y^2)y}{(x^2+y^2)^3}$$

$$r u \frac{\P v}{\P x} + r v \frac{\P v}{\P y} = -\frac{\P p}{\P y}.$$

$$\therefore \frac{\P p}{\P y} = -r \frac{10x}{x^2+y^2} \frac{-20xy}{(x^2+y^2)^2} - r \frac{10y}{x^2+y^2} \frac{10x^2 - 10y^2}{(x^2+y^2)^2} = r \frac{100(x^2+y^2)y}{(x^2+y^2)^3}$$

$$\therefore \vec{\nabla} p = \frac{\P p}{\P x}\hat{i} + \frac{\P p}{\P y}\hat{j} = \frac{100xr}{(x^2+y^2)^2}\hat{i} + \frac{100yr}{(x^2+y^2)^2}\hat{j} = \underline{\frac{100r}{(x^2+y^2)^2}}\ (x\hat{i} + y\hat{j}).$$

5.25 Check continuity (cylindrical coord from Table 5.1):

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = \frac{10}{r}\left(1 + \frac{1}{r^2}\right)\cos\theta + \frac{-10}{r}\left(1 + \frac{1}{r^2}\right)\cos\theta = 0.$$ $\therefore$ It is a possible flow. For Euler's Eqs. (let $n = 0$ in the momentum eqns of Table 5.1) in cylindrical coord:

$$\frac{\P p}{\P r} = r\frac{v_q^2}{r} - r v_r \frac{\P v_r}{\P r} - r\frac{v_q}{r}\frac{\P v_r}{\P q} = \frac{100r}{r}\left(1+\frac{1}{r^2}\right)^2 \sin^2 q - 10r\left(1 - \frac{1}{r^2}\right)\cos^2 q\left(\frac{20}{r^3}\right)$$
$$-\frac{10r}{r}\left(1+\frac{1}{r^2}\right)\sin^2 q\left(10 - \frac{10}{r^2}\right).$$

$$\frac{1}{r}\frac{\P p}{\P q} = -r\frac{v_r v_q}{r} - r v_r \frac{\P v_q}{\P r} - r\frac{v_q}{r}\frac{\P v_q}{\P q} = \frac{100r}{r}\left(1 - \frac{1}{r^4}\right)\sin q\cos q$$
$$-10r\left(1 - \frac{1}{r^2}\right)\cos q\sin q\left(\frac{20}{r^3}\right) - \frac{100r}{r}\left(1+\frac{1}{r^2}\right)^2 \sin q\cos q.$$

$$\therefore \vec{\nabla} p = \frac{\P p}{\P r}\hat{i}_r + \frac{1}{r}\frac{\P p}{\P q}\hat{i}_q = \underline{\frac{200r}{r^3}\left[\frac{1}{r^2} - \cos 2q\right]\hat{i}_r - \frac{200r}{r^3}\sin 2q\,\hat{i}_q}$$

5.26 This is an involved problem. Follow the steps of Problem 5.25. Good luck!

$$\frac{\partial p}{\partial r} = \rho\frac{\left(v_\theta^2 + v_\phi^2\right)}{r} - \rho v_r\frac{\partial v_r}{\partial r} - \rho\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = -\rho\frac{(v_r v_\theta)}{r} - \rho v_r\frac{\partial v_\theta}{\partial r} - \rho\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}$$

5.27 $\therefore\ \bar{p} = p - \left(\frac{2\mu}{3} + \lambda\right)\vec{\nabla}\cdot\vec{V}. \qquad \therefore\ \bar{p} - p = \underline{-\left(\frac{2\mu}{3} + \lambda\right)\vec{\nabla}\cdot\vec{V}}.$

$$\frac{\partial \hat{s}}{\partial s} \cong \frac{\Delta \hat{s}}{\Delta s} = -\frac{\Delta\alpha\hat{n}}{R\Delta\alpha} = -\frac{\hat{n}}{R}.$$

$$\frac{\partial \hat{s}}{\partial t} \cong \frac{\Delta \hat{s}}{\Delta t} = \frac{\hat{n}\ \Delta\theta}{\Delta t} = \hat{n}\frac{\partial\theta}{\partial t}.$$

$$\therefore\ \frac{D\vec{V}}{Dt} = \underline{\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}\right)\hat{s} + \left(V\frac{\partial \theta}{\partial t} - \frac{V^2}{R}\right)\hat{n}}.$$

For steady flow, the normal acc. is $\left(-\frac{V^2}{R}\right)$, the tangential acc. is $V\frac{\partial V}{\partial s}$.

5.28 For a rotating reference frame (see Eq. 3.2.15), we must add the terms due to $\vec{\Omega}$. Thus, Euler's equation becomes

$$\rho\left(\frac{D\vec{V}}{Dt} + 2\vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + \frac{d\vec{\Omega}}{dt}\times\vec{r}\right) = -\vec{\nabla}p - \rho\,\vec{g}.$$

5.29 $\tau_{xx} = -p + 2\mu\cancel{\frac{\partial u}{\partial x}} + \lambda\,\cancel{\vec{\nabla}\cdot\vec{V}} = -30 \text{ psi}.$

$\tau_{yy} = \tau_{zz} = -p = -30 \text{ psi}.$

$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right) = 10^{-5}\left[30 - 1440\times\frac{.1}{12}\right] = 18\times10^{-5} \text{ psf}.$

$\tau_{xz} = \tau_{yz} = 0. \qquad \frac{\tau_{xy}}{\tau_{xx}} = \frac{18\times10^{-5}}{30\times144} = \underline{4.17\times10^{-8}}.$

5.30 $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \frac{16y}{Cx^{9/5}} - \frac{16y^2}{C^2x^{13/5}}. \qquad \therefore\ v(x,y) = \frac{8y^2}{Cx^{9/5}} - \frac{16y^3}{3C^2x^{13/5}} + f(x).$

$v(x,o) = 0. \qquad \therefore\ f(x) = 0. \qquad 8 = C\ 1000^{4/5}. \qquad \therefore\ C = 0.0318.$

$\therefore\ u(x,y) = 629yx^{-4/5} - 9890y^2x^{-8/5}.$

$v(x,y) = 252y^2x^{-9/5} - 5270y^3x^{-13/5}.$

$\tau_{xx} = -p + 2\mu\frac{\partial u}{\partial x} = -100 + 0 = \underline{-100 \text{ kPa}}.$

$\tau_{yy} = \tau_{zz} = -p = \underline{-100 \text{ kPa}}.$

$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 2\times10^{-5}\left[629\times1000^{-4/5}\right] = \underline{5.01\times10^{-5} \text{ Pa}}.$

$\tau_{xz} = \tau_{yz} = 0.$

5.31 $$\frac{Du}{Dt} = \cancel{\frac{\partial u}{\partial t}} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) u = (\vec{V}\cdot\vec{\nabla})u.$$

$$\frac{Dv}{Dt} = \cancel{\frac{\partial v}{\partial t}} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) v = (\vec{V}\cdot\vec{\nabla})v.$$

$$\frac{Dw}{Dt} = \cancel{\frac{\partial w}{\partial t}} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) w = (\vec{V}\cdot\vec{\nabla})w$$

$$\therefore \frac{D\vec{V}}{Dt} = \frac{Du}{Dt}\hat{i} + \frac{Dv}{Dt}\hat{j} + \frac{Dw}{Dt}\hat{k} = \vec{V}\cdot\vec{\nabla}(u\hat{i} + v\hat{j} + w\hat{k}) = (\vec{V}\cdot\vec{\nabla})\vec{V}.$$

5.32 Follow the steps that lead to Eq. 5.3.17 and add the term due to compressible effects:

$$\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}p + \rho\vec{g} + \mu\nabla^2\vec{V} + \frac{\mu}{3}\frac{\partial}{\partial x}\vec{\nabla}\cdot\vec{V}\hat{i} + \frac{\mu}{3}\frac{\partial}{\partial y}\vec{\nabla}\cdot\vec{V}\hat{j} + \frac{\mu}{3}\frac{\partial}{\partial z}\vec{\nabla}\cdot\vec{V}\hat{k}$$

$$= -\vec{\nabla}p + \rho\vec{g} + \mu\nabla^2\vec{V} + \frac{\mu}{3}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)\vec{\nabla}\cdot\vec{V}$$

$$\therefore \underline{\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}p + \rho\vec{g} + \mu\nabla^2\vec{V} + \frac{\mu}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{V}).}$$

5.33 If $u=u(y)$, then continuity demands that $\frac{\partial v}{\partial y} = 0. \quad \therefore v = C.$

But, at $y=0$ (the lower plate) $v=0$. $\therefore C = 0$, and $v(x,y) = 0$.

$$\therefore \rho\frac{Du}{Dt} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

$$\therefore \underline{0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{a y^2}.}$$

$$\rho\frac{Dv}{Dt} = \underline{0 = -\frac{\partial p}{\partial y}.}$$

$$\rho\frac{Dw}{Dt} = 0 = -\frac{\partial p}{\partial z} + \rho(-g). \qquad \therefore \underline{0 = -\frac{\partial p}{\partial z} - \rho g.}$$

5.34 Continuity: $\frac{\partial}{\partial r}(rv_r) = 0. \quad \therefore rv_r = C.$ At $r = 0$, $v_r \neq \infty$. $\quad \therefore C = 0.$

$$\frac{Dv_r}{Dt} = \underline{0 = -\frac{1}{\rho}\frac{\partial p}{\partial r}.}$$

$$\frac{Dv_\theta}{Dt} = \underline{0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}.}$$

$$\rho\frac{Dv_z}{Dt}=\rho\left(\cancel{\frac{\partial v_z}{\partial t}}+\cancel{v_r}\frac{\partial v_z}{\partial r}+\frac{\cancel{v_\theta}}{r}\frac{\partial v_z}{\partial \theta}+v_z\cancel{\frac{\partial v_z}{\partial z}}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}}+\cancel{\frac{\partial^2 v_z}{\partial z^2}}\right)$$

$$\therefore\ \underline{0=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right).}$$

5.35 Continuity: $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)=0.$ $\therefore r^2 v_r=C.$ At $r=r_1,\ v_r=0.$ $\therefore C=0.$

$$\underline{-\frac{v_\theta^2}{r}\rho=-\frac{\partial p}{\partial r}+\mu\left(-\frac{2v_\theta}{r^2}\cot\theta\right).}$$

$$\underline{0=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2\sin^2\theta}\right]}$$

$$\underline{0=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}.}$$

5.36 For an incompressible flow $\vec{\nabla}\cdot\vec{V}=0$. Substitute Eqs. 5.3.10 into Eq. 5.3.2 and 5.3.3:

$$\rho\frac{Du}{Dt}=\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\rho g_x.$$

$$=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial^2 u}{\partial z^2}+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\rho g_x$$

$$\therefore\ \underline{\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x.}$$

$$\rho\frac{Dv}{Dt}=\frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(-p+2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\rho g_y.$$

$$=-\frac{\partial p}{\partial y}+\mu\frac{\partial^2 v}{\partial x^2}+\mu\frac{\partial^2 v}{\partial y^2}+\mu\frac{\partial^2 v}{\partial z^2}+\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\rho g_y$$

$$\therefore\ \underline{\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y.}$$

$$\rho\frac{Dw}{Dt}=\frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(-p+2\mu\frac{\partial w}{\partial z}\right)+\rho g_z$$

$$= -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial z^2} + \mu \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \rho g_z$$

$$\therefore \rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z .$$

5.37 If we substitute the constitutive equations (5.3.10) into Eqs. 5.3.2 and 5.3.3., with $\mu = \mu(x, y, z)$ we arrive at

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + 2\frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \mu}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial \mu}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

5.38 If plane flow is only parallel to the plate, $v = w = 0$. Continuity then demands that $\partial u / \partial x = 0$. The first equation of (5.3.14) simplifies to

$$\rho\left(\frac{\partial u}{\partial t} + u\cancel{\frac{\partial u}{\partial x}} + \cancel{v}\frac{\partial u}{\partial y} + \cancel{w}\frac{\partial u}{\partial z}\right) = -\cancel{\frac{\partial p}{\partial x}} + \rho \cancel{g_x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}$$

We assumed g to be in the y-direction, and since no forcing occurs other than due to the motion of the plate, we let $\partial p / \partial x = 0$.

5.39 From Eqs. 5.3.10, $-\frac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3} = p - \frac{2\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \lambda \vec{\nabla} \cdot \vec{V}$.

$$\therefore \bar{p} = p - \left(\frac{2\mu}{3} + \lambda\right)\vec{\nabla} \cdot \vec{V}. \qquad \therefore \bar{p} - p = -\left(\frac{2\mu}{3} + \lambda\right)\vec{\nabla} \cdot \vec{V}.$$

5.40 $(\vec{V} \cdot \nabla)\vec{V} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u\hat{i} + v\hat{j} + w\hat{k})$

$$\nabla \times (\vec{V} \cdot \nabla)\vec{V} = \left[\frac{\partial}{\partial y}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) - \frac{\partial}{\partial z}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\right]\hat{i}$$

$$+ \left[\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial x}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\right]\hat{k}$$

Use the definition of vorticity: $\vec{w} = (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\hat{i} + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\hat{j} + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\hat{k}$

$$(\vec{w}\cdot\nabla)\vec{V} = \left[(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\frac{\partial}{\partial x} + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\frac{\partial}{\partial y} + (\frac{\partial v}{\partial x} - \frac{\partial x}{\partial y})\frac{\partial}{\partial z} \right](u\hat{i} + v\hat{j} + w\hat{k})$$

$$(\vec{V}\cdot\nabla)\vec{w} = \left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right]\left[(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\hat{i} + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\hat{j} + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\hat{k} \right]$$

Expand the above, collect like terms, and compare coefficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$.

5.41 Studying the vorticity components of Eq. 3.2.21, we see that $w_z = -\P u/\P y$ is the only vorticity component of interest. The third equation of Eq. 5.3.24 then simplifies to

$$\frac{Dw_z}{Dt} = n\nabla^2 w_z$$

$$= n\frac{\P^2 w_z}{\P y^2}$$

since changes normal to the plate are much larger than changes along the plate, i.e., $\frac{\P w_z}{\P y} >> \frac{\P w_z}{\P x}$.

5.42 If viscous effects are negligible, as they are in a short section, Eq. 5.3.25 reduces to

$$\frac{Dw_z}{Dt} = 0$$

that is, there is no change in vorticity (along a streamline) between sections 1 and 2. Since (see Eq. 3.2.21), at section 1,

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -10$$

we conclude that, for the lower half of the flow at section 2,

$$\frac{\P u}{\P y} = 10.$$

This means the velocity profile at section 2 is a straight line with the same slope of the profile at section 1. Since we are neglecting viscosity, the flow can slip at the wall with a slip velocity u_0; hence, the velocity distribution at section 2 is $u_2(y) = u_0 + 10y$. Continuity then allows us to calculate the profile:

$$V_1 A_1 = V_2 A_2$$

$$\frac{1}{2}(10\times 0.04)(0.04w) = (u_0 + 10\times 0.02/\ 2)(0.02w). \qquad \therefore u_0 = 0.3 \text{ m / s}.$$

Finally,

$$u_2(y) = \underline{0.3 + 10y}$$

5.43 No. The first of Eqs. 5.3.24 shows that, neglecting viscous effects,

$$\frac{Dw_x}{Dt} = w_x \frac{\partial u}{\partial x} + w_y \frac{\partial u}{\partial y} + w_z \frac{\partial u}{\partial z}$$

so that w_y, which is nonzero near the snow surface, creates w_x through the term $w_y \partial u / \partial y$, since there would be a nonzero $\partial u / \partial y$ near the tree.

5.44 $$\int_{c.s.} k\vec{\nabla} T \cdot \hat{n} dA = \int_{c.v.} \frac{\partial}{\partial t}\left(\frac{V^2}{2} + gz + \tilde{u})\right)\rho d\mathcal{V} + \int_{c.s.}\left(\frac{V^2}{2} + gz + \tilde{u} + \frac{p}{\rho}\right)\rho \vec{V} \cdot \hat{n} dA$$

$$\int_{c.v.} \vec{\nabla} \cdot (k\vec{\nabla} T) d\mathcal{V} = \int_{c.v.} \frac{\partial}{\partial t}\left(\frac{V^2}{2} + gz + \tilde{u}\right)\rho d\mathcal{V} + \int_{c.v.} \vec{\nabla} \cdot \rho \vec{V}\left(\frac{V^2}{2} + gz + \tilde{u} + \frac{p}{\rho}\right) d\mathcal{V}$$

$$\therefore \int_{c.v.} \left[-k\nabla^2 T + \frac{\partial}{\partial t}\left(\rho \frac{V^2}{2} + \rho g z + \rho \tilde{u}\right) + \vec{\nabla} \cdot \rho \vec{V}\left(\frac{V^2}{2} + gz + \tilde{u} + \frac{p}{\rho}\right)\right] d\mathcal{V} = 0.$$

$$\frac{\partial}{\partial t}\rho \frac{V^2}{2} + \vec{\nabla} \cdot \rho \vec{V}\left(\frac{V^2}{2} + gz + \frac{p}{\rho}\right) = \frac{V^2}{2}\left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{V}\right) + \rho \vec{V} \cdot \left[\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} + \frac{\vec{\nabla} p}{\rho} + g\vec{\nabla} z\right] = 0.$$

continuity momentum

$$\therefore -k\nabla^2 T + \frac{\partial}{\partial t}\rho \tilde{u} + \rho \vec{V} \cdot \vec{\nabla} \tilde{u} = 0. \qquad \therefore \rho \frac{D\tilde{u}}{Dt} = k\nabla^2 T.$$

5.45 Divide each side by $dxdydz$ and observe that

$$\frac{\left.\frac{\partial T}{\partial x}\right|_{x+dx} - \left.\frac{\partial T}{\partial x}\right|_x}{dx} = \frac{\partial^2 T}{\partial x^2}, \qquad \frac{\left.\frac{\partial T}{\partial y}\right|_{y+dy} - \left.\frac{\partial T}{\partial y}\right|_y}{dy} = \frac{\partial^2 T}{\partial x^2}, \qquad \frac{\left.\frac{\partial T}{\partial z}\right|_{z+dz} - \left.\frac{\partial T}{\partial z}\right|_z}{dz} = \frac{\partial^2 T}{\partial z^2}$$

Eq. 5.4.5 follows.

5.46 $$\rho \frac{D\tilde{u}}{Dt} = \rho \frac{D(h - p/\rho)}{Dt} = \rho \frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{p}{\rho}\frac{D\rho}{Dt} = \rho \frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{p}{\rho}\left[-\rho \nabla \cdot \vec{V}\right]$$

where we used the continuity equation: $D\rho / Dt = -\rho \nabla \cdot \vec{V}$. Then Eq. 5.4. 9 becomes

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{p}{\rho}\left[-\rho \nabla \cdot \vec{V}\right] = K\nabla^2 T - p\nabla \cdot \vec{V}$$

which is simplified to

$$\rho \frac{Dh}{Dt} = K\nabla^2 T + \frac{Dp}{Dt}$$

5.47 See Eq. 5.4.9: $\tilde{u} = cT$. $\quad \therefore \rho c\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = k\nabla^2 T.$

Neglect terms with velocity: $\underline{\rho c\frac{\partial T}{\partial t} = k\nabla^2 T.}$

5.48 The dissipation function Φ involves viscous effects. For flows with extremely large velocity gradients, it becomes quite large. Then

$$r c_p \frac{DT}{Dt} = \Phi$$

and $\frac{DT}{Dt}$ is large. This leads to very high temperatures on reentry vehicles.

5.49 $u = 10(1 - 10\ 000\ r^2)$. $\quad \therefore \frac{\partial u}{\partial r} = -2r \times 10^5$. $\quad$ (r takes the place of y)

From Eq. 5.4.17, $\Phi = 2\mu\left[\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right] = \mu 4r^2 \times 10^{10}$.

At the wall where $r = 0.01$ m, $\quad \Phi = 1.8 \times 10^{-5} \times 4 \times .01^2 \times 10^{10} = \underline{72\ \text{N/m}^2 \cdot \text{s}}$.

At the centerline $\frac{\partial u}{\partial r} = 0$ so $\Phi = \underline{0}$.

At a point half-way: $\Phi = 1.8 \times 10^{-5} \times 4 \times .005^2 \times 10^{10} = \underline{18\ \text{N/m}^2 \cdot \text{s}}$.

5.50 (a) Momentum: $\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$

Energy: $\rho c\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2$.

(b) Momentum: $\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} + \frac{\partial \mu}{\partial y}\frac{\partial u}{\partial y}$

Energy: $\rho c\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2$.